Chapter 3
Modifed Wave Number Analysis

T. H. Pulliam / NASA Ames

Modified Wave Number Analysis

- Arbitrary periodic functions can be decomposed into their Fourier components, which are in the form $e^{i\kappa x}$, where κ is the wavenumber. For a general κ

$$u(x) = c_\kappa e^{i\kappa x}$$

- The exact derivative in x

$$\frac{\partial\, u(x)}{\partial x} = i\kappa c_\kappa e^{i\kappa x} = i\kappa\, u(x)$$

- How will a finite-difference operator δ_x approximate the derivative of $u_j = c_\kappa e^{i\kappa x_j}$, $x_j = j\Delta x$

- By definition: ($i\kappa^*$ is defined to be modified wave number)

$$\delta_x u_j = i\kappa^* c_\kappa e^{i\kappa x_j} = i\kappa^* u_j$$

- The particular form of $i\kappa^*$ depends on the choice of δ_x

- *Note: We define $i\kappa^*$ as the modified wave number, leaving in the i. As we shall see below $i\kappa^*$ can be complex, i.e., have a real and imaginary part, the significance of which will become clear later*

Modified Wave Number - Central Differencing

- Central Difference:

$$\delta_x^c \, u_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

- Using $u_j = e^{i\kappa j \Delta x}$ we have

$$\delta_x^c u_j = \frac{e^{i\kappa(j+1)\Delta x} - e^{i\kappa(j-1)\Delta x}}{2\Delta x} =$$

$$\frac{e^{\,i\kappa\Delta x} - e^{-\,i\kappa\Delta x}}{2\Delta x} e^{i\kappa j\Delta x}$$

$$= \frac{e^{\,i\kappa\Delta x} - e^{-\,i\kappa\Delta x}}{2\Delta x} u_j = i\kappa_c^* u_j$$

- Using the definition of the complex exponential $e^{i\kappa\Delta x} = cos(\kappa\Delta x) + isin(\kappa\Delta x)$ we have

$$i\kappa_c^* = \ i\frac{sin(\kappa\Delta x)}{\Delta x}$$

- Modified wave number $i\kappa^*$ is an approximation to $i\kappa$.

- For δ_x^c, using the infinite series expansion of $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$i\ \frac{sin(\kappa\Delta x)}{\Delta x} = \frac{i}{\Delta x}\left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right] = \\ i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right] \quad (1)$$

- Therefore $i\kappa_c^* = i\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4) = i\kappa + O(\Delta x^2)$, a second order approximation.

1^{st} Order Backward Differencing

- Backward Difference:

$$\delta_x^b \, u_j = \frac{u_j - u_{j-1}}{\Delta x}$$

- Using $u_j = e^{i\kappa j \Delta x}$ we have

$$\begin{aligned} \delta_x^b u_j = \frac{e^{i\kappa j \Delta x} - e^{i\kappa (j-1) \Delta x}}{\Delta x} &= \frac{1 - e^{-\,i\kappa \Delta x}}{\Delta x} e^{i\kappa j \Delta x} = \\ \frac{1 - e^{-\,i\kappa \Delta x}}{\Delta x} u_j &= i\kappa_b^* u_j \end{aligned}$$

- Expanding in sin and cos

$$i\kappa_b^* = \frac{1 - cos(\kappa \Delta x) + isin(\kappa \Delta x)}{\Delta x}$$

- For δ_x^b, using the infinite series expansion of $sin(x)$ and $cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$$
\begin{aligned}
i\kappa_b^* &= \frac{1}{\Delta x}\left[\frac{(\kappa\Delta x)^2}{2} + O(\Delta x^4)\right] \\
&+ \frac{1}{\Delta x} i \left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right] \\
&= \frac{\kappa^2\Delta x}{2} + O(\Delta x^3) + i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right]
\end{aligned}
$$

- Therefore $i\kappa_b^* = i\kappa + O(\Delta x)$ a first order approximation.

Modified Wave Number: Compact Schemes

- Modified Wave Number analysis can be also applied to the class of compact schemes
- Need to add definition:

$$\delta_x u_{j+m} = i\kappa^* e^{m \cdot i\kappa\Delta x} e^{i\kappa x_j} c_\kappa = i\kappa^* e^{m \cdot i\kappa\Delta x} u_j$$

- Fourth-order Padé scheme

$$(\delta_x u)_{j-1} + 4(\delta_x u)_j + (\delta_x u)_{j+1} = \frac{3}{\Delta x}(u_{j+1} - u_{j-1})$$

- The modified wavenumber for this scheme satisfies

$$i\kappa^* e^{-i\kappa\Delta x} + 4i\kappa^* + i\kappa^* e^{i\kappa\Delta x} = \frac{3}{\Delta x}(e^{i\kappa\Delta x} - e^{-i\kappa\Delta x})$$

- Collecting terms

$$i\kappa^* = \frac{3i \sin \kappa\Delta x}{(2 + \cos \kappa\Delta x)\Delta x}$$

- Series expansion shows a 4^{th} Order scheme.

Solution to the Discrete PDE

- The discrete PDE is

$$\frac{\partial u(t)_j}{\partial t} + a\delta_x u(t)_j = 0$$

- Using separation of variables: $u(t)_j = e^{i\kappa j \Delta x} f(t)$ and applying the general result $\delta_x u_j = i\kappa^* u_j$

$$\frac{\partial e^{i\kappa j \Delta x} f(t)}{\partial t} + a i\kappa^* e^{i\kappa j \Delta x} f(t) = 0$$

- The ODE for $f(t)$ is $\frac{\partial f(t)}{\partial t} + af(t)i\kappa^* = 0$ with solution $f(t) = f(0)e^{-ai\kappa^*\, t}$ giving

$$u(t)_j = c_\kappa e^{i\kappa j \Delta x} e^{-ai\kappa^*\, t}, \quad c_\kappa = f(0)$$

Exact Solution - Discrete Solution

- From Above we have:

$$u(t)_j = c_\kappa e^{i\kappa j \Delta x} e^{-ai\kappa^*\, t}, \quad c_\kappa = f(0)$$

- Comparing this discrete solution with the continuous solution

$$u(x,t) = c_\kappa e^{i\kappa x} e^{-ai\kappa\, t}$$

- The difference between κ and κ^* shows how the choice of δ_x affects the phase and amplitude of the computed solution.

Effect of Modified Wave

- Central differencing $i\kappa_c^* = i\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)$, then

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\ t} =$$

$$c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right]\ t}$$

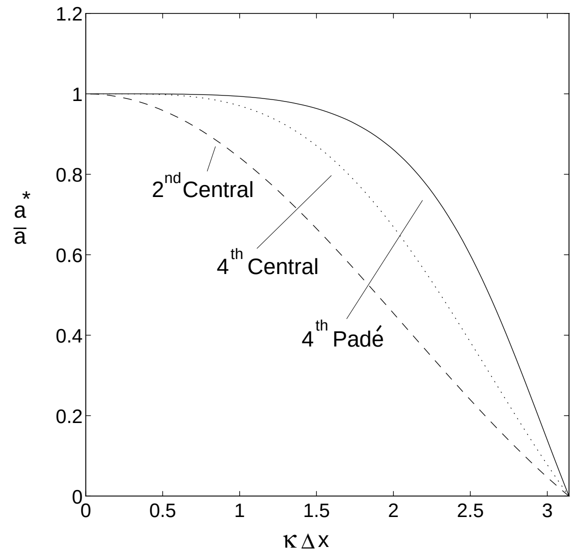
- Dropping the $O(\Delta x^4)$ term and defining $a^* = a\left[1 - \frac{(\kappa\Delta x)^2}{6}\right]$ the modified wave speed

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-a^* i\kappa\ t}$$

- This shows that each wave slows down by $\frac{(\kappa\Delta x)^2}{6}$ which is a function of κ.

Effect on Phase Speed

- Define $\frac{a^*}{a} = \frac{\kappa^*}{\kappa}$
- For second order central differencing $\frac{a^*}{a} = \frac{\sin \kappa\Delta x}{\kappa\Delta x}$



Effect of Modified Wave: Complex $i\kappa^*$

- Following the same reasoning for the backward differencing $i\kappa_b^*$

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-a\left[\frac{\kappa^2\Delta x}{2} + i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6}\right]\right]t}$$

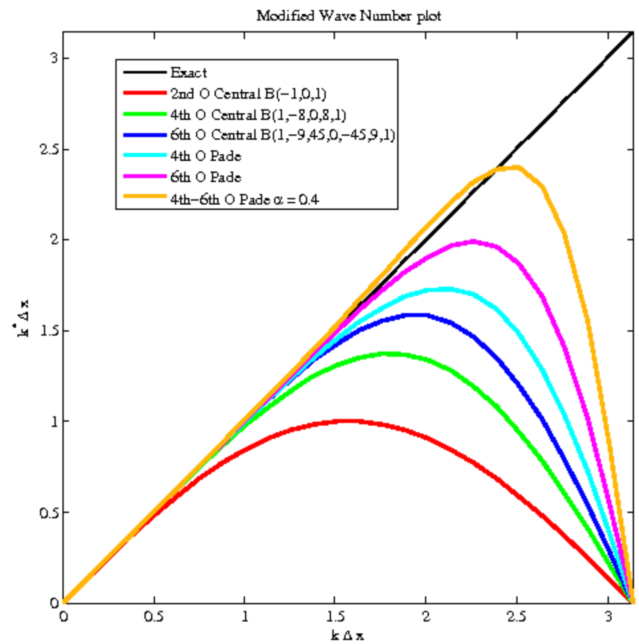
- Slowing down the waves and also damping them by $\frac{\kappa^2\Delta x}{2}$
- *Note: The Modified Wave Number is complex and contributes to both the phase and amplitude errors*

Modified Wave Number

- Exact PDE solution:$u(x,t) = c_\kappa e^{i\kappa x} e^{-ai\kappa\ t}$
- Discrete Solution: $u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa^*\ t}$
- The $Imag(i\kappa^*) - \kappa$ represents dispersion, phase, frequency error
- The $Real(i\kappa^*)$ creates a amplification error.
- Note: $|e^{-ai\kappa t}| = 1$, no amplitude change for exact PDE solution
- From discrete solution: $|e^{-ai\kappa^* t}| = |e^{Real(-ai\kappa^* t)}|$
  - If $Real(-ai\kappa^* t) < 0$: Decay for $a > 0, t > 0$
  - If $Real(-ai\kappa^* t) > 0$: Growth for $a > 0, t > 0$
- To assess amplitude error plot $|e^{Real(-i\kappa^*)}|$

Modified Wave Number: Centered Schemes

- 2^{nd} Order Central: $B(-1,0,1)/(2\Delta x)$
- 4^{th} Order Central: $B(1,-8,0,8,-1)/(12\Delta x)$
- 6^{th} Order Central: $B(-1,9,-45,0,45,-9,1)/(30\Delta x)$
- $4^{th}-6^{th}$ Order Centered Compact: $B(1,\alpha,1)^{-1}B(-\beta,-2\phi,,0,2\phi,\beta)/(4\Delta x)$ with $\phi = \frac{4+2\alpha}{3}$ and $\beta = \frac{4-\alpha}{3}$



Modified Wave Number: Upwind Schemes

- 1^{st} Order : $B(-1,1,0)/(\Delta x)$
- 2^{nd} Order : $B(1,-4,3,0,0)/(2\Delta x)$
- 3^{rd} Order : $B(1,-6,3,2,0)/(6\Delta x)$